Marks: 25	FYJC Subject: Mathematics I Continuity	Time: 1 hr.

Q.1. Attempt any three out of five. **(9Marks)**

1. $f(x) = \frac{\sqrt{x+3}-2}{x^3-1}$ for $x \neq 1$

$= 2$ for $x = 1$, at $x = 1$.

Ans. $f(1) = 2$ (Given)… (1)

$$\lim_{x\to1} f(x) = \lim_{x\to1} \frac{\sqrt{x+3}-2}{x^3-1}$$

$$= \lim_{x\to1} \frac{\sqrt{x+3}-2}{x^3-1} \times \frac{\sqrt{x+3}+2}{\sqrt{x+3}+2}$$

$$= \lim_{x\to1} \frac{(x+3)-4}{(x^3-1)(\sqrt{x+3+2})}$$

$$= \lim_{x\to1} \frac{x-1}{(x-1)(x^2+x+1)(\sqrt{x+3+2})}$$

$$= \lim_{x\to1} \frac{1}{(x^2+x+1)(\sqrt{x+3+2})}$$

$[\therefore x \to 1, x \neq 1 \quad \therefore x-1 \neq 0]$

$$= \frac{\lim_{x\to1} 1}{\lim_{x\to1}(x^2+x+1) \times = \lim_{x\to1}(\sqrt{x \times 3}+2}$$

$$= \frac{1}{(1+1+1)(\sqrt{1+3}+2}$$

$$= \frac{1}{12}$$ …… (2)

From (1) and (2),

$$= \lim_{x\to1} f(x) \neq f(1)$$

$\therefore f\ is$ discontinuous at $x = 1$.

2. If $f(x) = \left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$ for $x \neq 0$,

is continuous at $x = 0$ then find $f(0)$.

Ans. Given that $f(x)$ is continuous at $x = 0$

$$\therefore f(0) = \lim_{x\to0} f(x)$$

$$f(0) = \lim_{x\to0} \left(\frac{3x+2}{2-5x}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to0} \left(\frac{2\left(1+\frac{3x}{2}\right)}{2\left(1-\frac{5x}{2}\right)}\right)^{\frac{1}{x}}$$

$$= \lim_{x\to0} \left(\frac{\left(1+\frac{3x}{2}\right)^{\frac{1}{x}}}{\left(1-\frac{5x}{2}\right)^{\frac{1}{x}}}\right)$$

$$= \frac{\left[\lim_{x\to0}\left(1+\frac{3x}{2}\right)^{\frac{2}{3x}}\right]^{\frac{3}{2}}}{\left[\lim_{x\to0}\left(1+\frac{5x}{2}\right)^{\frac{-2}{5x}}\right]^{\frac{-5}{2}}}$$

$$= \frac{e^{\frac{3}{2}}}{e^{\frac{-5}{2}}}$$

$$= e^{\frac{3}{2}+\frac{5}{2}} = e^{\frac{8}{2}} = e^4 \because \left[\lim_{x\to 0}(1+kx)^{\frac{1}{kx}} = e\right]$$

3. $f(x) = x^3 - 2x + 1,\quad$ for $x \leq 2$

$= 3x - 2,\quad$ for $x > 2$, at $x = 2$.

Ans.

$f(x) = x^3 - 2x + 1$, if $x \leqslant 2$

$\therefore \lim_{x\to 2^-} f(x) = \lim_{x\to 2}(x^3 - 2x + 1)$

$= 2^3 - 2(2) + 1$

$= 5$

Also, $f(x) = 3x - 2$, if $x > 2$

$\therefore \lim_{x\to 2^+} f(x) = \lim_{x\to 2}(3x - 2)$

$= 3(2) - 2$

$= 4$

$\because \lim_{x\to 2^-} f(x) \neq \lim_{x\to 2^+} f(x)$

$\therefore \lim_{x\to 2} f(x)$ does not exist

$\therefore f$ is **discontinuous** at $x = 2$.

4. if $f(x) = \frac{5^x + 5^{-x} - 2}{x^2}\quad$ for $x \neq 0$

$= k\quad$ for $x = 0$

is continuous at $x = 0$,find k

Ans.

$f(0) = k$ ... (Given) ... (1)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{5^x + 5^{-x} - 2}{x^2}$$

$$= \lim_{x\to 0} \frac{5^x(5^x + 5^{-x} - 2)}{5^x \cdot x^2}$$

$$= \lim_{x\to 0} \frac{(5^x)^2 + 1 - 2\cdot 5^x}{5^x \cdot x^2}$$

$$= \lim_{x\to 0} \frac{(5^x - 1)^2}{5^x \cdot x^2} = \lim_{x\to 0}\left(\frac{5^x - 1}{x}\right)^2 \cdot \frac{1}{5^x}$$

$$= \left(\lim_{x\to 0} \frac{5^x - 1}{x}\right)^2 \times \frac{1}{\lim_{x\to 0} 5^x}$$

$$= (\log 5)^2 \times \frac{1}{5^0} \quad \ldots \left[\because \lim_{x\to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$= (\log 5)^2 \quad \ldots (2)$$

Since f is continuous at $x = 0$,

$f(0) = \lim_{x\to 0} f(x)$

$\therefore k = (\log 5)^2$ ... [By (1) and (2)]

5. $f(x) = \frac{x^2+18x-19}{x-1}$ for $x \neq 1$

$= 20$ for $x = 1$, at $x = 1$

Ans.

$f(1) = 20$ ... (Given) ... (1)

$$\lim_{x\to 1} f(x) = \lim_{x\to 1} \frac{x^2+18x-19}{x-1}$$

$$= \lim_{x\to 1} \frac{(x-1)(x+19)}{x-1}$$

$$= \lim_{x\to 1} (x+19)$$

... $[\because x \to 1, x \neq 1, \therefore x-1 \neq 0]$

$= 1 + 19$

$= 20$ ... (2)

From (1) and (2),

$$\lim_{x\to 1} f(x) = f(1)$$

$\therefore f$ is **continuous** at $x = 1$.

Q.2. Attempt any four out of six. **(16Marks)**

1. $f(x) = \frac{45^x - 9^x - 5^x + 1}{(k^x-1)(3^x-1)}$ for $x \neq 0$

$= \frac{2}{3}$ for $x = 0$, at $x = 0$

Ans.

$f(0) = \frac{2}{3}$ ... (Given) ... (1)

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{45^x - 9^x - 5^x + 1}{(k^x-1)(3^x-1)}$$

$$= \lim_{x\to 0} \frac{9^x(5^x-1)-(5^x-1)}{(k^x-1)(3^x-1)}$$

$$= \lim_{x\to 0} \frac{(5^x-1)(9^x-1)}{(k^x-1)(3^x-1)}$$

$$= \lim_{x\to 0} \frac{\left(\frac{5^x-1}{x}\right)\left(\frac{9^x-1}{x}\right)}{\left(\frac{k^x-1}{x}\right)\left(\frac{3^x-1}{x}\right)}$$

... $[\because x \to 0 \ \therefore x \neq 0]$

$$= \frac{\left[\lim_{x\to 0} \frac{5^x-1}{x}\right] \times \left[\lim_{x\to 0} \frac{9^x-1}{x}\right]}{\left[\lim_{x\to 0} \frac{k^x-1}{x}\right] \times \left[\lim_{x\to 0} \frac{3^x-1}{x}\right]}$$

$$= \frac{(\log 5) \times (\log 9)}{(\log k) \times (\log 3)} \quad \ldots \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a \right]$$

$$= \frac{(\log 5) \times (2 \log 3)}{(\log k) \times (\log 3)} = \frac{2 \log 5}{\log k} \quad \ldots (2)$$

Now, f is continuous at $x = 0$

$$\therefore f(0) = \lim_{x \to 0} f(x)$$

$$\therefore \frac{2}{3} = \frac{2 \log 5}{\log k} \quad \ldots \text{[By (1) and (2)]}$$

$$\therefore \log k = 3 \log 5 = \log 5^3 = \log 125$$

$$\therefore k = 125.$$

2. $f(x) = \frac{x^3 - 8}{\sqrt{x+2} - \sqrt{3x-2}}$ for $x \neq 2$

$= -24$ for $x = 2$, at $x = 2$

Ans.

$f(2) = -24$ ... (Given) ... (1)

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x+2} - \sqrt{3x-2}}$$

$$= \lim_{x \to 2} \frac{x^3 - 8}{\sqrt{x+2} - \sqrt{3x-2}} \times \frac{\sqrt{x+2} + \sqrt{3x-2}}{\sqrt{x+2} + \sqrt{3x-2}}$$

$$= \lim_{x \to 2} \frac{(x^3 - 8)[\sqrt{x+2} + \sqrt{3x-2}]}{(x+2) - (3x-2)}$$

$$= \lim_{x \to 2} \frac{(x-2)(x^2 + 2x + 4)[\sqrt{x+2} + \sqrt{3x-2}]}{-2(x-2)}$$

$$= \lim_{x \to 2} \frac{(x^2 + 2x + 4)[\sqrt{x+2} + \sqrt{3x-2}]}{-2}$$

$$\ldots [\because x \to 2,\ x \neq 2 \quad \therefore x - 2 \neq 0]$$

$$= -\frac{1}{2}\left[\left\{ \lim_{x \to 2} (x^2 + 2x + 4) \right\} \times \left\{ \lim_{x \to 2} (\sqrt{x+2} + \sqrt{3x-2}) \right\} \right]$$

$$= -\frac{1}{2}[(4 + 4 + 4) \times (\sqrt{2+2} + \sqrt{6-2})]$$

$$= -24 \quad \ldots (2)$$

From (1) and (2),

$$\lim_{x \to 2} f(x) = f(2)$$

$\therefore f$ is continuous at $x = 2$

3. $f(x) = \dfrac{\log x - \log 3}{x-3}$ for $x \neq 3$

$= 3$ for $x = 3$, at $x = 3$.

Ans.

$f(3) = 3$ ... (Given) ... (1)

$$\lim_{x \to 3} f(x) = \lim_{x \to 3} \frac{\log x - \log 3}{x - 3}$$

Put $x - 3 = h$ $\therefore$ as $x \to 3$, $h \to 0$

Also, $x = 3 + h$

$$\therefore \lim_{x \to 3} f(x) = \lim_{h \to 0} \frac{\log(3+h) - \log 3}{h}$$

$$= \lim_{h \to 0} \frac{\log\left(\frac{3+h}{3}\right)}{h}$$

$$= \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{3}\right)}{\left(\frac{h}{3}\right)} \times \frac{1}{3}$$

$$= \frac{1}{3} \lim_{h \to 0} \frac{\log\left(1 + \frac{h}{3}\right)}{\left(\frac{h}{3}\right)}$$

$$= \frac{1}{3} \times 1 \quad \ldots \left[\because h \to 0,\ \frac{h}{3} \to 0 \text{ and } \lim_{x \to 0} \frac{\log(1+x)}{x} = 1\right]$$

$$= \frac{1}{3} \quad \ldots (2)$$

From (1) and (2),

$$\lim_{x \to 3} f(x) \neq f(3)$$

$\therefore$ f **is discontinuous** at $x = 3$.

4. $f(x) = \left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$ for $x \neq 2$

$= \text{k}$ for $x = 2$, at $x = 2$.

Ans.

$f(2) = k$ ... (Given) ... (1)

$$\lim_{x \to 2} f(x) = \lim_{x \to 2} \left(\frac{5x-8}{8-3x}\right)^{\frac{3}{2x-4}}$$

Put $x = 2 + h$ $\therefore$ as $x \to 2$, $h \to 0$

$$\therefore \lim_{x \to 2} f(x) = \lim_{h \to 0} \left[\frac{5(2+h)-8}{8-3(2+h)}\right]^{\frac{3}{2(h+2)-4}}$$

$$= \lim_{h \to 0} \left[\frac{2+5h}{2-3h}\right]^{\frac{3}{2h}}$$

$$= \lim_{h \to 0} \left[\frac{1+\frac{5h}{2}}{1-\frac{3h}{2}}\right]^{\frac{3}{2h}}$$

$$= \lim_{h \to 0} \frac{\left(1+\frac{5h}{2}\right)^{\frac{3}{2h}}}{\left(1-\frac{3h}{2}\right)^{\frac{3}{2h}}}$$

$$= \frac{\lim_{h \to 0} \left(1+\frac{5h}{2}\right)^{\frac{3}{2h}}}{\lim_{h \to 0} \left(1-\frac{3h}{2}\right)^{\frac{3}{2h}}}$$

$$= \frac{\left[\lim_{h \to 0} \left(1+\frac{5h}{2}\right)^{\frac{2}{5h}}\right]^{\frac{15}{4}}}{\left[\lim_{h \to 0} \left(1-\frac{3h}{2}\right)^{-\frac{2}{3h}}\right]^{-\frac{9}{4}}}$$

$$= \frac{e^{\frac{15}{4}}}{e^{-\frac{9}{4}}}$$

$$\ldots \left[\begin{array}{l} \because h \to 0 \quad \therefore \frac{5h}{2} \to 0, \; -\frac{3h}{2} \to 0 \\ \text{and } \lim_{x \to 0} (1+x)^{\frac{1}{x}} = e \end{array}\right]$$

$$= e^{\frac{15}{4}+\frac{9}{4}}$$

$$= e^{6} \quad \ldots (2)$$

Now, f is continuous at $x = 2$

$\therefore f(2) = \lim_{x \to 2} f(x)$

$\therefore k = e^6$ ... [By (1) and (2)]

5. $f(x) = \frac{32^x - 1}{8^x - 1} + a$, for $x \neq 2$

$= 2$, for $x = 0$

$= x + 5 - 2b$, for $x < 0$

Is continuous at $x = 0$

Ans.

$f(0) = 2$ ... (Given) ... (1)

$f(x) = \frac{32^x - 1}{8^x - 1} + a$, for $x > 0$

$$\lim_{x \to 0^+} f(x) = \lim_{x \to 0} \left[\frac{32^x - 1}{8^x - 1} + a\right]$$

$$= \lim_{x \to 0} \frac{32^x - 1}{8^x - 1} + \lim_{x \to 0} a$$

$$= \lim_{x \to 0} \frac{\left(\frac{32^x - 1}{x}\right)}{\left(\frac{8^x - 1}{x}\right)} + a \quad \ldots [\because x \to 0 \therefore x \neq 0]$$

$$= \frac{\log 32}{\log 8} + a \quad \ldots \left[\because \lim_{x \to 0} \frac{a^x - 1}{x} = \log a\right]$$

$$= \frac{5 \log 2}{3 \log 2} + a$$

$$= \frac{5}{3} + a \quad \ldots (2)$$

Also, $f(x) = x + 5 - 2b$, for $x < 0$

$$\therefore \lim_{x \to 0^-} f(x) = \lim_{x \to 0} (x + 5 - 2b)$$

$$= 0 + 5 - 2b$$

$$= 5 - 2b \quad \ldots (3)$$

Now, f is continuous at $x = 0$

$$\therefore f(0) = \lim_{x \to 0^+} f(x) = \lim_{x \to 0^-} f(x)$$

$$\therefore 2 = \frac{5}{3} + a = 5 - 2b \quad \ldots \text{[By (1), (2) and (3)]}$$

$$\therefore a + \frac{5}{3} = 2 \text{ and } 2 = 5 - 2b$$

$$\therefore a = 2 - \frac{5}{3} = \frac{1}{3}$$

and $2b = 3$ $\therefore b = \frac{3}{2}$

Hence, $a = \frac{1}{3}$ and $b = \frac{3}{2}$

6. $f(x) = \frac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$ for $x \neq 2$

$= \frac{1}{5}$ for $x = 2,$ at $x = 2.$

Ans.

$f(2) = \frac{1}{5}$ ... (Given) ... (1)

$\lim_{x \to 2} f(x) = \lim_{x \to 2} \frac{\sqrt{x-1}-(x-1)^{\frac{1}{3}}}{x-2}$

Put $x = 1 + h$. Then as $x \to 2$, $h \to 1$

Also $x - 1 = h$ and $x - 2 = (1+h) - 2 = h - 1$

$\therefore \lim_{x \to 2} f(x) = \lim_{h \to 1} \frac{\sqrt{h}-h^{\frac{1}{3}}}{h-1}$

$= \lim_{h \to 1} \frac{(\sqrt{h}-1)-(h^{\frac{1}{3}}-1)}{h-1}$

$= \lim_{h \to 1} \left[\frac{\sqrt{h}-1}{h-1} - \frac{h^{\frac{1}{3}}-1}{h-1}\right]$

$= \lim_{h \to 1} \left(\frac{h^{\frac{1}{2}}-1^{\frac{1}{2}}}{h-1}\right) - \lim_{h \to 1} \left(\frac{h^{\frac{1}{3}}-1^{\frac{1}{3}}}{h-1}\right)$

$= \frac{1}{2}(1)^{-\frac{1}{2}} - \frac{1}{3}(1)^{-\frac{2}{3}}$ ... $\left[\because \lim_{x \to a} \frac{x^n - a^n}{x-a} = na^{n-1}\right]$

$= \frac{1}{2} - \frac{1}{3}$

$= \frac{1}{6}$ ... (2)

From (1) and (2),

$\lim_{x \to 2} f(x) \neq f(2)$

$\therefore f$ is **discontinuous** at $x = 2$